A note on the upper bound for multi-color Ramsey number of odd cycles *

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Abstract

Let $r_k(G)$ be the k-color Ramsey number of a graph G, and let C_{2m+1} be an odd cycle of length $2m+1$. In this note, we prove that for fixed $m \geq 3$,

$$r_k(C_{2m+1}) < c^{k-1}\big(k!\big)^{1/2+\delta}$$

for all $k \geq 3$, where $\delta = 1/(4m^3 - 8m^2 + 8m - 2)$ and $c = c(m) > 0$ is a constant. This result improves an old result by Bondy and Erdős (Ramsey numbers for cycles in graphs, J. Combin. Theory Ser. B 14 (1973) 46-54).

Keywords: Ramsey number; odd cycle; upper bound

1 Introduction

Let G be a graph. The multi-color *Ramsey number* $r_k(G)$ is defined as the minimum integer N such that each edge coloring of the complete graph K_N with k colors containing a monochromatic G. It is easy to see that if G is a bipartite graph with $ex(N;G) \leq (c + o(1))\frac{1}{2}n^{2-1/t}$, then $r_k(G) \leq (1+o(1))(ck)^t$, where $ex(N;G)$ is the Turán number of G, i.e. the maximum number of edges among all graphs of order N that contain no G. Thus $r_k(G)$ is bounded from above by a polynomial if G is a bipartite graph. The situation, however, becomes dramatically different when G is non-bipartite. Let C_{2m+1} be an odd cycle of length $2m+1$. It was shown that

$$1073^{k/6} \leq r_k(C_3) \leq c \cdot k!,$$

where $c > 0$ is a constant, see [2, 5, 7, 9, 14, 15]. For general odd cycles, Bondy and Erdős [4] obtained

$$m2^k + 1 \leq r_k(C_{2m+1}) \leq (2m+1)(k+2)!. \tag{1}$$

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The upper bound in (1) was improved slightly by Graham et al. [10] to $r_k(C_{2m+1}) < 2m(k+2)!$. For $m = 2$, Li [11] showed that $r_k(C_5) \le c\sqrt{18^k k!}$ for all $k \ge 3$, where $0 < c < 1/10$ is a constant.

In this note, we prove the following upper bound for $r_k(C_{2m+1})$.

Theorem 1 *Let $m \ge 3$ be a fixed integer. We have*

$$r_k(C_{2m+1}) < c^{k-1}\big(k!\big)^{1/2+\delta}$$

for all $k \ge 3$, where $\delta = 1/(4m^3 - 8m^2 + 8m - 2)$ and $c = c(m) > 0$ is a constant.

Remark. Let $N = r_k(G) - 1$. From the definition, there exists a k-edge coloring of K_N containing no monochromatic G. In such an edge coloring, any graph induced by a monochromatic set of edges is called a Ramsey graph. There is some evidence supporting a claim that Ramsey graphs for $r_k(G)$ are nearly regular, as one can see [1, 3, 13] for various known Ramsey colorings and random graphs.

Let $\epsilon > 0$ be a constant. Under the assumption that each Ramsey graph H for $r_k(C_{2m+1})$ has minimum degree at least $\epsilon d(H)$ for large k, Li [11, Theorem 2] showed that $r_k(C_{2m+1}) \le \big(c^k k!\big)^{1/m}$, where $d(H)$ is the average degree of H and $c = c(\epsilon, m) > 0$ is a constant. Under such an assumption, which we do not know how to prove, the upper bound for $r_k(C_{2m+1})$ is much better than that in Theorem 1.

2 Proof of the main result

In order to prove Theorem 1, we need the following results.

Theorem 2 *(Chvátal [6]) Let T_m be a tree of order m. We have*

$$r(T_m, K_n) = (m-1)(n-1) + 1.$$

For a graph G, let $\alpha(G)$ denote the independence number of G.

Lemma 1 *(Li and Zang [12]) Let $m \ge 2$ be an integer and let $G = (V, E)$ be a graph of order N that contains no C_{2m+1}. We have*

$$\alpha(G) \ge \frac{1}{(2m-1)2^{(m-1)/m}} \left(\sum_{v \in V} d(v)^{1/(m-1)} \right)^{(m-1)/m}.$$

Proof of Theorem 1. Let $m \ge 3$ and $k \ge 3$ be integers. For convenience, denote $r_k = r_k(C_{2m+1})$ and $N = r_k - 1$. Let $K_N = (V, E)$ be the complete graph on vertex set V of order N. From the definition, there exists an edge-coloring of K_N using k colors such that it contains no monochromatic C_{2m+1}. Let E_i denote the monochromatic set of edges in color i for $i = 1, 2, \ldots, k$. Without loss of generality, we may assume that E_1 has the largest

cardinality among all $E_i{}'s$. Therefore $|E_1| \geq \binom{N}{2}/k$. Let G be the graph with vertex set V and edge set E_1. Then the average degree d of G satisfies

$$d = \frac{2|E_1|}{N} \geq \frac{N-1}{k} = \frac{r_k - 2}{k}.$$

Consider an independent set I of G with $|I| = \alpha(G)$. Since any edge of K_N between two vertices in I is colored by one of the colors $2, 3, \ldots, k$, the subgraph induced by I is an edge-colored complete graph using $k-1$ colors, which contains no monochromatic C_{2m+1}. Thus $|I| \leq r_{k-1} - 1$. Let us denote

$$a = \frac{1}{2} + \delta, \tag{2}$$

where $\delta = 1/(4m^3 - 8m^2 + 8m - 2)$.

Claim. We have that

$$\alpha(G) > \frac{N}{9m} k^{-a}.$$

Proof. We shall separate the proof of the claim into two cases.

Case 1. There exists a vertex $v \in V$ with $d(v) \geq D$, where $D = Nk^{-a}$.

Note that there is no path of $2m$ vertices in the neighborhood of v since otherwise together with v it would form a C_{2m+1}, which is a contradiction. Hence, Theorem 2 implies that

$$\alpha(G) \geq \frac{D}{2m-1} > \frac{D}{2m} = \frac{N}{2m} k^{-a}.$$

Case 2. The degree of each vertex is less than D.

Set

$$p_1 = -\frac{1}{2} - \frac{1}{2(m-1)} + \frac{m\delta}{m-1}, \quad \text{and}$$
$$p_2 = -\frac{1}{2} - \frac{1}{2(m-1)^2} + \frac{m^2\delta}{(m-1)^2}.$$

Recall the definition of a in (2). It is not difficult to verify that

$$1 + p_1 = (m-1)(p_2 - p_1), \quad 1 + p_2 = -(m-1)(a + p_1), \tag{3}$$

and

$$p_1(m-1) = m(a-1). \tag{4}$$

Now, let us partition the vertex set V in terms of the magnitude of the degrees. Let x_1 be the number of vertices whose degrees are between $d/2$ and Nk^{p_1}, let x_2 be the number of vertices whose degrees are between Nk^{p_1} and Nk^{p_2}, and let x_3 be the number of vertices whose degrees are between Nk^{p_2} and D. Then

$$Nd = \sum_{v \in V} d(v) \leq x_1 Nk^{p_1} + x_2 Nk^{p_2} + x_3 D + \left(N - \sum_{i=1}^{3} x_i\right) \times \frac{d}{2},$$

which implies that

$$x_1 N k^{p_1} + x_2 N k^{p_2} + x_3 D \geq \frac{Nd}{2}. \tag{5}$$

Let

$$y = x_1 \left(\frac{N-1}{2k}\right)^{1/(m-1)} + x_2 \big(N k^{p_1}\big)^{1/(m-1)} + x_3 \big(N k^{p_2}\big)^{1/(m-1)}.$$

From the inequality (5), we have

$$\begin{aligned}
y &\geq \left(\frac{d}{2} k^{-p_1} - x_2 k^{p_2 - p_1} - x_3 k^{-a-p_1}\right)\left(\frac{N-1}{2k}\right)^{1/(m-1)} + x_2 \big(N k^{p_1}\big)^{1/(m-1)} + x_3 \big(N k^{p_2}\big)^{1/(m-1)} \\
&= \frac{d}{2}\left(\frac{N-1}{2k}\right)^{1/(m-1)} k^{-p_1} + x_2 \left[\big(N k^{p_1}\big)^{1/(m-1)} - \left(\frac{N-1}{2k}\right)^{1/(m-1)} k^{p_2-p_1}\right] \\
&\quad + x_3 \left[\big(N k^{p_2}\big)^{1/(m-1)} - \left(\frac{N-1}{2k}\right)^{1/(m-1)} k^{-a-p_1}\right].
\end{aligned}$$

Note that from (3), we have

$$\frac{\big(N k^{p_1}\big)^{1/(m-1)}}{(N/k)^{1/(m-1)} k^{p_2-p_1}} = k^{(1+p_1)/(m-1)-p_2+p_1} = 1,$$

and

$$\frac{\big(N k^{p_2}\big)^{1/(m-1)}}{(N/k)^{1/(m-1)} k^{-a-p_1}} = k^{(1+p_2)/(m-1)+a+p_1} = 1.$$

Thus,

$$y \geq \frac{d}{2}\left(\frac{N-1}{2k}\right)^{1/(m-1)} k^{-p_1}.$$

Therefore, by applying Lemma 1 and $d \geq (N-1)/k$, we have that for all $k \geq 3$,

$$\begin{aligned}
\alpha(G) &> \frac{1}{4m}\left(\sum_{v \in V} d(v)^{1/(m-1)}\right)^{(m-1)/m} \geq \frac{1}{4m}\left[\frac{d}{2}\left(\frac{N-1}{2k}\right)^{1/(m-1)} k^{-p_1}\right]^{(m-1)/m} \\
&\geq \frac{1}{4m}\left[\frac{N-1}{2k}\left(\frac{N-1}{2k}\right)^{1/(m-1)} k^{-p_1}\right]^{(m-1)/m} = \frac{1}{8m} \cdot \frac{N-1}{k^{1+p_1(m-1)/m}} \\
&> \frac{1}{9m} N k^{-1-p_1(m-1)/m},
\end{aligned}$$

where the last inequality holds since $N = r_k(C_{2m+1}) - 1 \geq m 2^k$ from (1).

Thus, by noticing that $p_1(m-1) = m(a-1)$ from (4), we have

$$\alpha(G) > \frac{1}{9m} N k^{-1-p_1(m-1)/m} = \frac{1}{9m} N k^{-a}.$$

The proof of the claim is completed. $\square$

Now, note that $|I| \le r_{k-1} - 1$ and the above claim, we have that

$$r_{k-1} - 1 \ge \frac{1}{9m} k^{-a} (r_k - 1).$$

Faudree and Schelp [8] proved that $r_2(C_{2m+1}) = 4m + 1$ for $m \ge 2$. Repeated application of the above bound yields that

$$\begin{aligned} r_k - 1 &\le 9m \cdot k^a \cdot (r_{k-1} - 1) \\ &\le (9m)^{k-2} \big(k \cdot \cdots \cdot 3\big)^a \big(r_2(C_{2m+1}) - 1\big) \\ &= (9m)^{k-2} \big(k!\big)^a \cdot 2^{-a} \cdot 4m. \end{aligned}$$

Since $2^{-a} < 1$, we can conclude that

$$r_k - 1 < (9m)^{k-1} \big(k!\big)^a.$$

This completes the proof of Theorem 1. $\square$

Finally, let us propose the following problem.

Problem 1 *Let $m \ge 2$ be an integer. Prove or disprove that $r_k(C_{2m+1}) = o((k!)^{1/m})$ as $k \to \infty$.*

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